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A GENERALIZED CAP MODEL FOR GEOLOGICAL MATERIALS

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The full scope and theoretical basis of the cap model is described so as to demonstrate and define the capabilities and limitations of the model. A cap model falls within the framework of the classical incremental theory of plasticity and is based on a loading function which serves as both a yield surface and plastic potential. The model describes material behavior in compression and, in general, some type of tension behavior must be postulated for completeness. Tensile behavior will not

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20. ABSTRACT (Continued).

be considered here. Particular forms of the cap model are adequate for many purposes, however, it is desirable to describe the model in its most general form so as to clearly indicate the adaptability and flexibility as well as the limits of the cap model approach. This has been done, and the procedure for fitting the cap model is briefly described.

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I INTRODUCTION

In two previous studies, Ref. [1] and [2], particular cap models for soils and rocks were introduced. The purpose of this report is to indicate the manner in which previous models may be generalized, if necessary, to more adequately describe the behavior of geological materials. The full scope and theoretical basis of the cap model is described so as to demonstrate and define the capabilities and limits of the model. Also, the fitting procedure and use of a particular version of the general model is illustrated by means of an example.

From a general point of view, a cap model falls within the framework of the classical incremental theory of plasticity and is based on a loading function which serves as both a yield surface and plastic potential. Typically, the loading function is assumed to be isotropic and to consist of two parts: a modified Drucker-Prager, Ref. [3], yield condition, denoted by

$$f_1(J_1 \, , \, J_2') = 0 \tag{1}$$

in which J_1 and J_2' are the first and second invariants of the stress and deviatoric stress tensors, respectively, together with hardening plastic cap

$$f_2(J_1 \, , \, J_2' \, , \, \kappa) = 0 \tag{2}$$

which may expand or contract as the hardening parameter κ increases or decreases. These are illustrated in Fig. 1. The model describes material behavior in compression ($J_1 \leq 0$)

and, in general, some type of tension behavior must be postulated for completeness. Tensile behavior (for which $J_1 > 0$) will not be considered here.

In the previous models for soils, the hardening parameter was taken to be

$$\kappa = \varepsilon_v^p \tag{3}$$

in which ε_v^p is the plastic volumetric strain, while for rocks

$$\kappa = \int_0^t -f_1(J_1\ ,\ J_2')\left[(\dot{\varepsilon}_1^p)^2 + (\dot{\varepsilon}_2^p)^2 + (\dot{\varepsilon}_3^p)^2\right]dt \tag{4}$$

in which the $\dot{\varepsilon}_i^p$ are the principal components of the plastic strain rate tensor and t is time. The use of Eq. (3) permitted the cap to reverse itself when a point on the yield curve $f_1 = 0$ was reached, thus controlling the excessive dilatancy predicted for soils by the Prager-Drucker model. The use of Eq. (4) for rocks, which does not permit the cap to move back, ensures dilatancy while permitting hysteresis in a hydrostatic load - unload cycle. Such volumetric hysteresis is not present in the Prager-Drucker model.

In the earlier models, the elastic portion of the behavior (which is most important in determining the unloading and reloading behavior of the model) was generally assumed to be linear, i.e., it was described by constant bulk and shear moduli.

It was found that good fits (well within the scatter of the experimental data) of stress-strain curves and loading paths were obtained using an exponential function for f_1 and an ellipse (tangent to $f_1 = 0$ for rocks, and with a horizontal tangent at its intersection with $f_1 = 0$ for soils) for $f_2 = 0$.

While these particular forms of cap model are adequate for many purposes, it is desirable to describe the model in its most general form so as to clearly indicate the adaptability and flexibility as well as the limits of the cap model approach. This is done in Section II. In Section III, the procedure for fitting the cap model is briefly described, while Section IV gives an example of a particular form of cap model which was used in ground shock calculations.

II THE GENERALIZED CAP MODEL

The classical theory of plasticity allows for a broad range of material behavior, and the cap model falls within this range. Many previous applications of plasticity theory for metals involved the assumption that volumetric strains are purely elastic. The cap model however, is predicated on the fact that the volumetric hysteresis exhibited by many geologic materials can also be described by a plasticity model, if the model is based on a hardening yield surface which includes conditions of hydrostatic stress. Guidelines as to how this may be done have been provided by Drucker, Ref. [4], whose stability postulate is sufficient, although not necessary, to satisfy all thermodynamic and continuity requirements for continuum models. Stability ensures that all physically reasonable initial-boundary value problems are properly posed in the mathematical sense.

The basic implications of Ref. [4] with respect to the plastic portion of the material behavior are:

1) The yield surface should be convex in stress space
2) The loading function and plastic potential should coincide (associated flow rule)
3) Plastic strain or work "softening" should not occur, i.e.,

$$\dot{\sigma}_{ij} \, \dot{\varepsilon}^{p}_{ij} \geq 0 \tag{5}$$

in which σ_{ij} and ε^{p}_{ij} are the components of the stress and plastic strain tensors.

These conditions allow considerable leeway in choosing the

functional forms f_1 and f_2 in Eqs. (1, 2), but the particular functions given in Section IV seem capable of adequately describing most geological materials.

It should be noted that work by Mroz, Ref. [5] and Bleich, Ref. [6], suggest that it may be possible to violate the above conditions in certain cases without destroying the stability of the model.

The fitting of stress-strain curves and loading paths using a cap model can be improved by introducing a nonlinear elastic component of behavior. This has been done by replacing the constant bulk and shear moduli in the linear elastic stress increment-strain increment equations

$$dJ_1 = 3K\ dI \tag{6}$$

$$ds_{ij} = 2G\ de_{ij} \tag{7}$$

by

$$K = K(J_1) \tag{8}$$

and

$$G = G(J_2') \tag{9}$$

In the above equations I is the first invariant of the strain tensor and s_{ij} and e_{ij} are the stress and strain deviators, respectively.

Additional flexibility in fitting experimental data can be introduced into the model by generalizing Eqs. (8) and (9) to

$$K = K\ (J_1, \kappa) \tag{10}$$

$$G = G\ (J_2', \kappa) \tag{11}$$

It should be noted that Eqs. (10) and (11), or Eqs. (8) and (9), correspond to a hypoelastic model, Ref. [7], with a positive definite elastic internal energy, W, which is independent of stress path. This may be shown by writing

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \int_0^{\sigma_{ij}} (s_{ij} + \frac{1}{3} J_1 \delta_{ij})(\frac{ds_{ij}}{2G} + \frac{dJ_1}{9K} \delta_{ij})$$

$$= \int_0^{s_{ij}} \frac{1}{2G} s_{ij} ds_{ij} + \int_0^{J_1} \frac{1}{9K} J_1 dJ_1 \tag{12}$$

$$= \int_0^{J_2'} \frac{dJ_2'}{2G(J_2', \kappa)} + \int_0^{J_1} \frac{d(J_1^2)}{18K(J_1, \kappa)}$$

During elastic deformation κ is constant, so that the integrals in the last member of Eq. (12) depend only on the current values of J_2' and J_1 as well as on κ. Therefore, W is independent of path during purely elastic deformations. Further, since G and K are always positive, as is J_2', W is positive definite. Therefore, there is no possibility of energy generation by the model.

In problems involving wave propagation at sites consisting of layers of both soil and rock, it is desirable to be able to use the same digital computer program for layers of both kinds. This is not possible when the different hardening parameters of Eqs. (3) and (4) are used. If, instead of Eq. (4), the hardening parameter for rocks is taken to be

the maximum previous value of plastic compaction,

$$\kappa = \varepsilon^{p}_{v(max)} \tag{13}$$

a generalized program which may be used for both soils and rocks can be easily written. It should be pointed out that the use of Eq. (13) instead of Eq. (4) does not permit the introduction of dilatancy until the stress point first reaches the yield surface $f_1 = 0$, but this loss in accuracy of the fit may be less important than the improved efficiency of using one generalized model for all geological materials.

It is also noteworthy that in those problems in which no advantage is gained by using essentially the same computer code for rock as that for soil, and where the rock is not too porous, a model incorporating a yield surface $f_1 = 0$, no cap, and different bulk moduli for loading and unloading, is sufficiently accurate and simpler to use than any cap model. This possibility is discussed in more detail in Appendix A.

It is also possible to include other features in cap models, such as anistropy, rate dependence, and hardening of the modified Drucker-Prager portion of the yield surface (isotropic and/or kinematic hardening). Further, in some cases it may be desirable to replace hypoelastic behavior with hyperelastic behavior, Ref. [7], within the yield surface. Since these additional possibilities have not been sufficiently studied, they will not be included in this paper, but will

be reported upon later.

If it is necessary to introduce finite strains, Jaumann's stress rate and the rate of deformation replace the stress and strain rates of Eqs. (6) and (7), Ref. [7].

III PROCEDURE FOR FITTING OF THE CAP MODEL

The procedure for obtaining the functional forms and parameters used in cap models constructed for use in ground shock computations is based on representative material data obtained from laboratory tests on material samples. Recently, in-situ material tests have come to be recognized as having important potential in ascertaining material behavior in situations of importance in ground shock computations. Work is currently in progress along several fronts to determine methods by which such tests can be incorporated into the overall procedure for determining material behavior.

The representative material behavior generally consists of uniaxial strain and triaxial stress data. Sometimes, hydrostatic, proportional loading, direct shear and/or other tests are available. The first step in the fitting procedure is to employ the unloading portion of these tests to determine appropriate elastic behavior of the model, since the cap model behaves elastically during initial unloading in these tests. For example, as long as the model behaves elastically, unloading behavior indicates the bulk modulus K in hydrostatic tests, the shear modulus G in triaxial stress tests, and the combination K + 4/3 G in uniaxial stress tests. Other tests, if available, may be used to check or adjust the overall fit.

The next step in the fitting procedure is to establish the failure envelope, i.e., the portion of the yield surface

which limits the shearing stresses that the material can withstand. While the failure envelope could be chosen as a work or strain hardening yield surface, it is generally adequate and much simpler to assume it to be ideally plastic. The failure envelope is generally obtained using failure data from triaxial stress and proportional loading tests. This data is fit by a function of the stresses, and is usually assumed to involve only the first stress invariant and the second invariant of the stress deviators.

The remaining step in the fitting procedure is the most difficult. The cap portion of the model is obtained by a trial and error procedure in which a cap shape and hardening rule are assumed and the behavior of this assumed model is computed and compared to the representative material data. If the fit requires improvement, a new set of parameters is tried and the procedure is repeated. The computation of the model behavior can be based on the equations describing the relations between the stress and strain increments during the common laboratory loading paths. These equations are derived in Appendix B for the case of uniaxial strain.

Obviously, the success of such a trial and error procedure and the rapidity with which it converges is strongly dependent on the experience of the modeler. Knowledge of the effect on the model behavior of changes in the cap model parameters is important for rapidly obtaining a satisfactory fit. For example, the fitting procedure is

greatly simplified by the knowledge (obtained through experience in fitting several cap models by trial and error) that the hardening rule strongly affects the stress-strain curves for uniaxial strain and hydrostatic paths, while the shape of the cap plays an important role in determining the stress-strain behavior for triaxial stress situations and the stress path for uniaxial strain. In fact, the hardening rule has been obtained for the most recent cap models by using a separate program to compute the plastic volumetric strain during hydrostatic loading.

IV AN EXAMPLE OF A GENERALIZED CAP MODEL

In this section an example of the type of cap model used in recent ground shock calculations is presented. Specific model parameters and the resulting model behavior as well as the original material properties to which the model was fitted are also presented.

The failure envelope and cap, Fig. 1, which describe the entire yield surface of the material, intersect at $J_1 = L$ The complete yield function, $F(J_1, \sqrt{J_2'}, \varepsilon_v^p) = 0$, is

$$F(J_1, \sqrt{J_2'}, \varepsilon_v^p) = \begin{cases} F_1(J_1, \sqrt{J_2'}) = \sqrt{J_2'} - [A - Ce^{BJ_1}] = 0 \quad \text{if } L < J_1 & (14) \\ F_2(J_1, \sqrt{J_2'}, \varepsilon_v^p) = (J_1 - L)^2 + R^2 J_2' - (X - L)^2 & (15) \\ \qquad = 0 \quad \text{if } L > J_1 & \end{cases}$$

in which A, B and C are material constants. The two quantities L and X are functions of ε_v^p and represent respectively the values of J_1 at the center of the cap and at the intersection of the cap with the J_1-axis. These two quantities are related to each other through the parameter ℓ by

$$\ell - X = R\,[A - Ce^{B\ell}]$$

$$L = \begin{cases} \ell \text{ if } \ell \leq 0 \\ 0 \text{ if } \ell > 0 \end{cases} \tag{16}$$

and are related to the plastic volumetric strain ε_v^p by means of the hardening function

$$\varepsilon_v^p = W[e^{DX} - 1 - \alpha DXe^{(D_1X - D_2X^2)}] - W_F X^2 e^{D_F X} \tag{17}$$

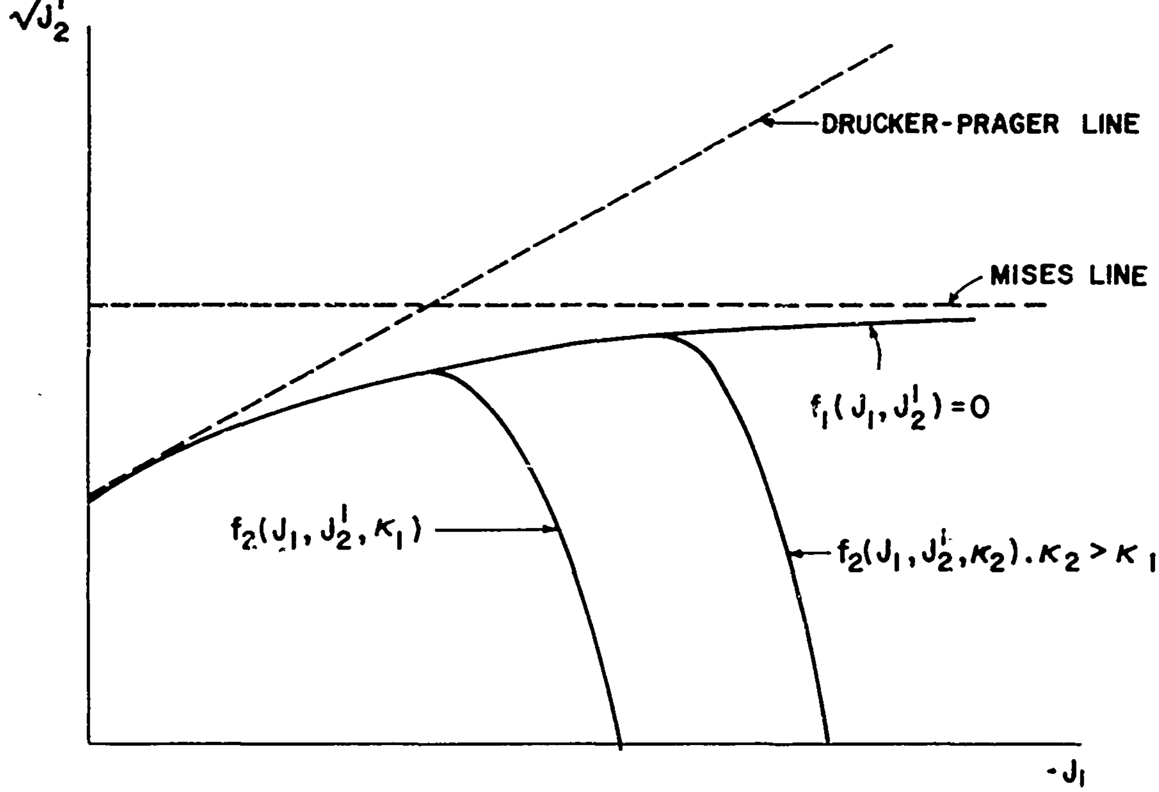

FIG. 1 LOADING FUNCTION FOR CAP MODEL

in which W, D, α, D_1 , D_2 , W_F and D_F are material constants. The quantity R in Eqs. (15, 16) represents the ratio of the major to the minor axis of the cap and is given by

$$R(L) = \frac{R_0}{1+R_1}\left[1 + R_1 e^{R_2 L}\right] + R_3 e^{-R_4 (L+R_5)^2} \tag{18}$$

in which R_0, R_1, R_2, R_3, R_4 and R_5 are material constants.

The elastic portion of the model behavior is represented by the bulk and shear moduli

$$K(J_1, \varepsilon_v^p) = K_{EI}\,\frac{1 + \delta e^{K_1 J_1}}{1 + \beta e^{K_1 J_1}} + \frac{K_s}{2 \cosh(K_2 \varepsilon_v^p)} \tag{19}$$

$$G(J_2', \varepsilon_v^p) = G_{EI}\,\frac{e^{G_1\sqrt{J_2'}} + \gamma e^{-G_2\sqrt{J_2'}}}{1 + \eta e^{-G_2\sqrt{J_2'}}} + \frac{G_s}{2 \cosh(G_3 \varepsilon_v^p)} \tag{20}$$

in which K_{EI}, β, δ, K_1, K_2, K_s, γ, G_{EI}, η, G_1, G_2, G_3 and G_s are material constants.

For the material constants listed below the model behavior is shown in Figs. 2-4 together with the material representative properties to which the model was fitted. In general, the model agrees quite well with the representative properties. The model parameters are

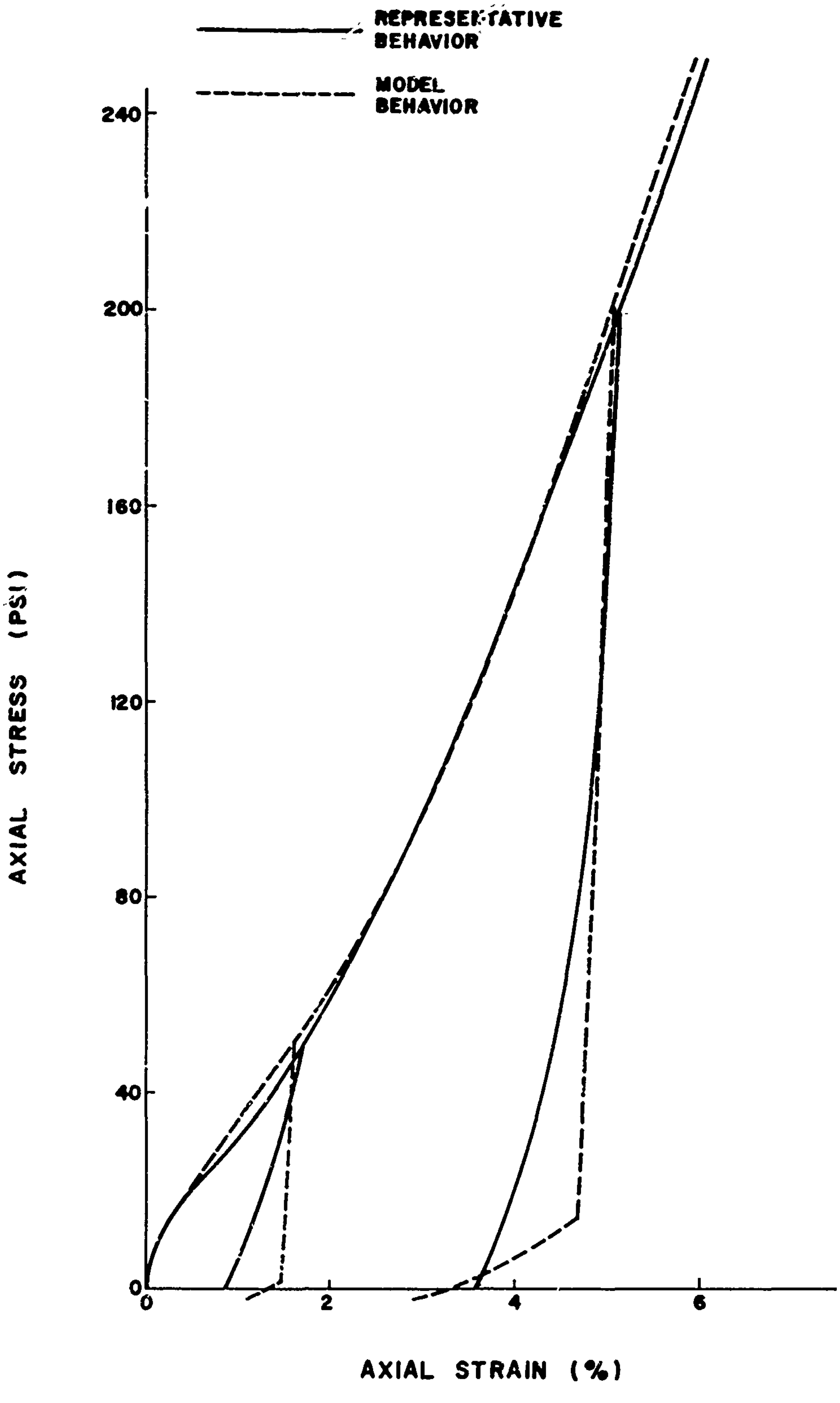

FIG. 2 STRESS-STRAIN BEHAVIOR OF MODEL VS. REPRESENTATIVE MATERIAL IN UNIAXIAL STRAIN

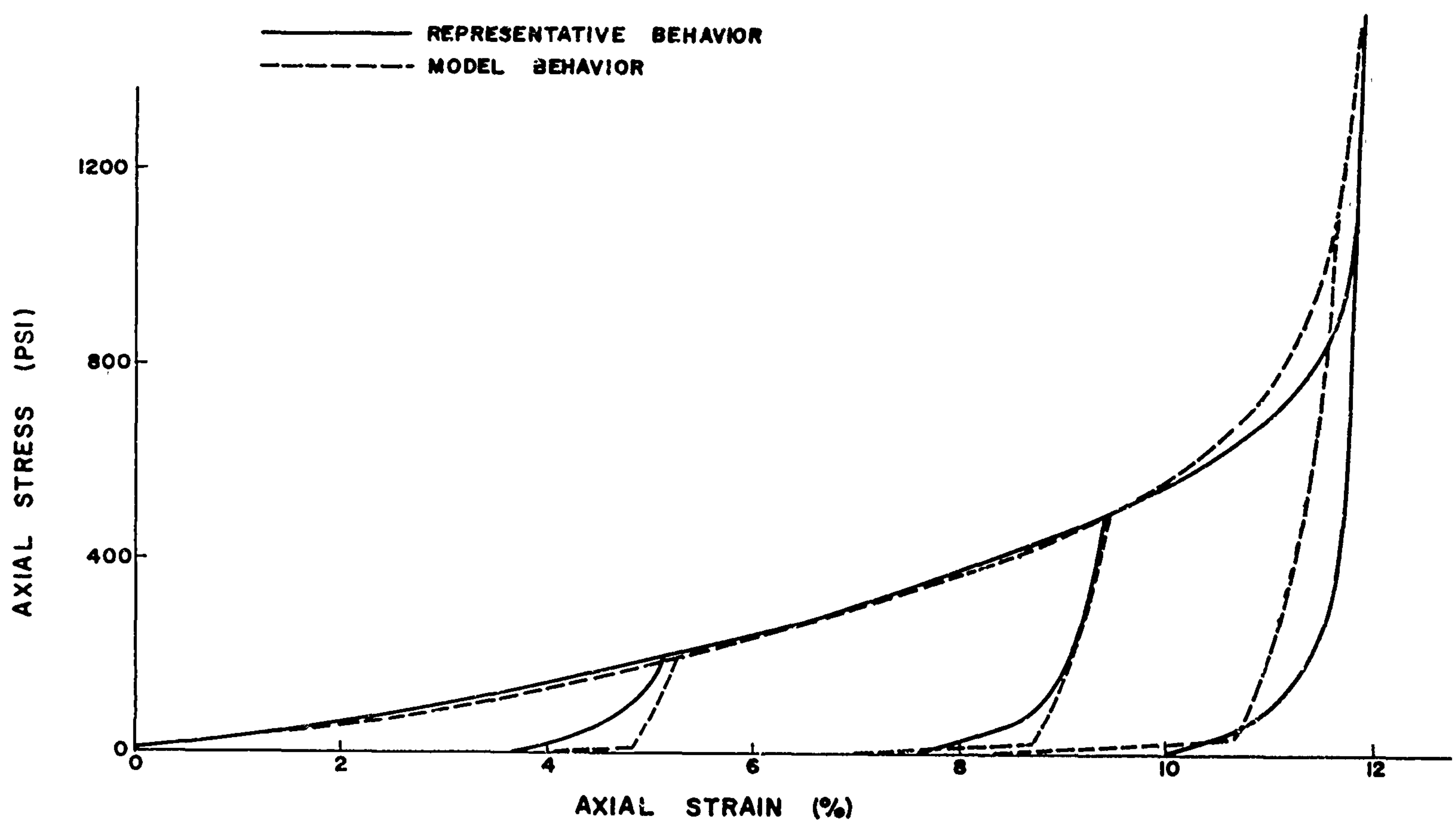

FIG. 3 STRESS STRAIN BEHAVIOR OF MODEL VS. REPRESENTATIVE MATERIAL IN UNIAXIAL STRAIN

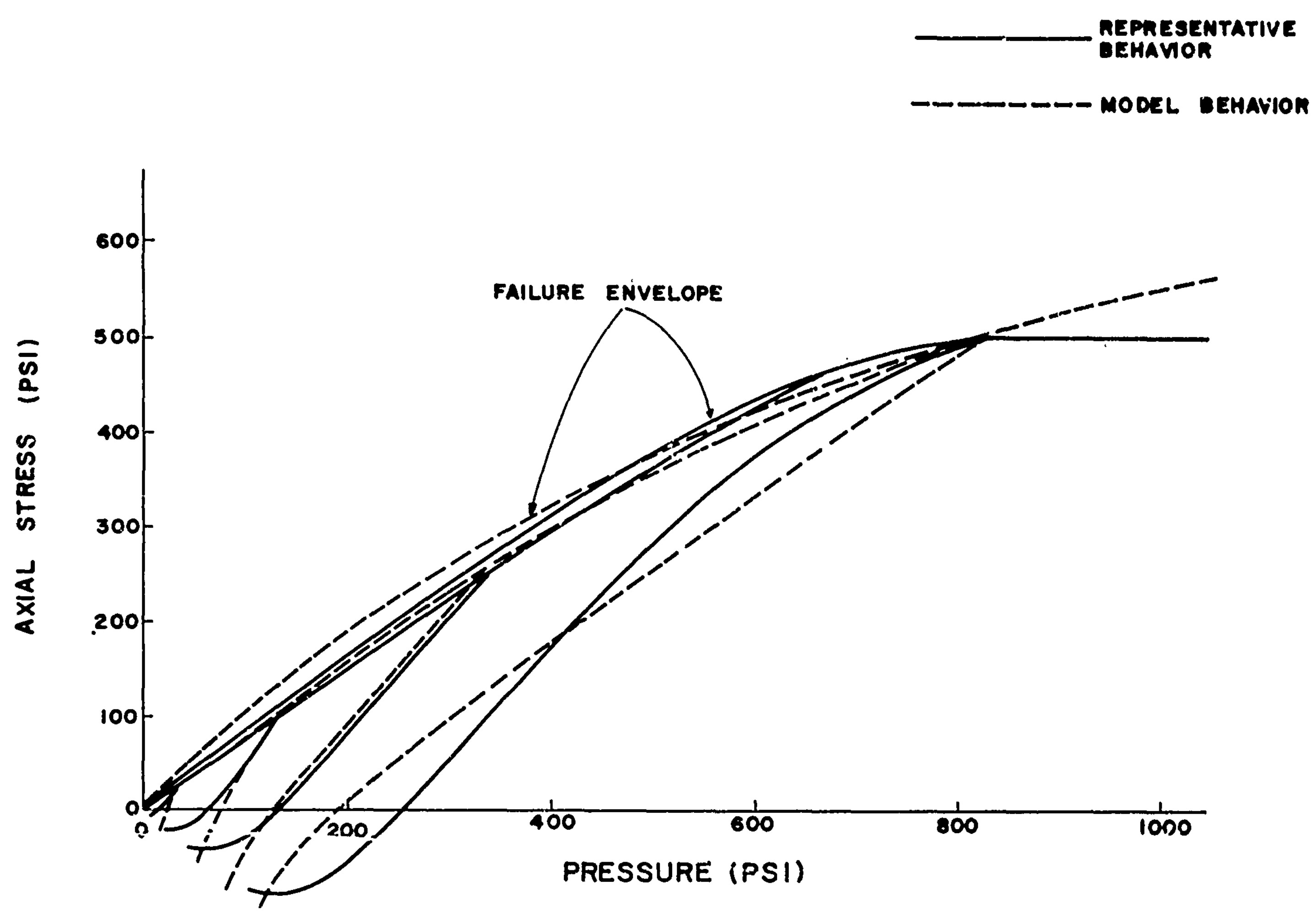

FIG. 4 STRESS PATH BEHAVIOR OF MODEL VS. REPRESENTATIVE MATERIAL IN UNIAXIAL STRAIN

$A = 0.405$ ksi	$R_4 = 0.03$ ksi^{-2}
$B = 0.5$ ksi^{-1}	$R_5 = 7.0$ ksi
$C = 0.403$ ksi	$K_{EI} = 444.$ ksi
$W = 0.101$	$K_s = 26.6$ ksi
$D = 2.0$ ksi^{-1}	$\beta = 5.28$
$\alpha = 1.0$	$\delta = -0.924$
$D_1 = 16.0$ ksi^{-1}	$K_1 = 0.693$ ksi^{-1}
$D_2 = 0.0$ ksi^{-2}	$K_2 = 80$
$W_F = -0.43$ ksi^{-1}	$G_{EI} = 267$ ksi
$D_F = 3.65$ ksi^{-1}	$G_s = 5.4$ ksi
$R_0 = 4.3$	$\eta = 4$
$R_1 = 0.8$	$G_1 = 0.0$ ksi^{-1}
$R_2 = 4.0$ ksi^{-1}	$G_2 = 3.0$ ksi^{-1}
$R_3 = -3.5$	$G_3 = 200$

$$\gamma = -0.7$$

It should be noted that, in general, the choice of material parameters cannot be made in completely arbitrary fashion. For example,

$$A \geq C \tag{22}$$

$$B \geq 0 \tag{23}$$

$$R \geq 0 \tag{24}$$

so the parameters R_0, R_1, R_2, R_3, R_4 and R_5 must be chosen so as to satisfy Eq. (24) for all possible states of the material. Further, the loading condition in conjunction with the yield condition $F_2 = 0$ requries

$$\frac{\partial F_2}{\partial \varepsilon_v^p} d\varepsilon_v^p = - \frac{\partial F_2}{\partial \sigma_{ij}} d\sigma_{ij} \leq 0 \tag{25}$$

and $K > 0$ and $G > 0$ must also be satisfied for all achievable stress states of the material.

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APPENDIX A

It has been found while making fits for rocks of low porosity, that the cap, $f_2 = 0$, may be chosen as essentially vertical except near its intersection with the yield surface $f_1 = 0$. The behavior of a cap model with a vertical cap and a bulk modulus K (which may be a constant or function of pressure) which is the same for loading and unloading (see Fig. 5), i.e.,

$$K_L = K_U \tag{A-1}$$

is identical to the uncapped model (see Fig. 6) with

$$K_L < K_U \tag{A-2}$$

That this is so can be readily seen by noticing that if an associated flow rule is applied to a vertical cap only plastic volume changes occur. The resulting hysteresis in the stress-strain curves is also produced without the cap if Eq. (A-2) is used.

It should be noted that use of an inequality like Eq. (A-2) involving the shear modulus would violate the continuity requirements that load paths infinitesimally close to a neutral loading path should result in essentially the same stress-strain curves.

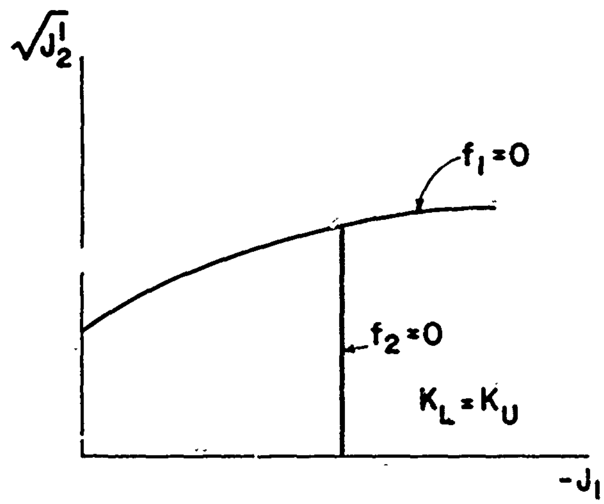

FIG. 5 CAP MODEL WITH SAME BULK MODULUS ON LOADING AND UNLOADING

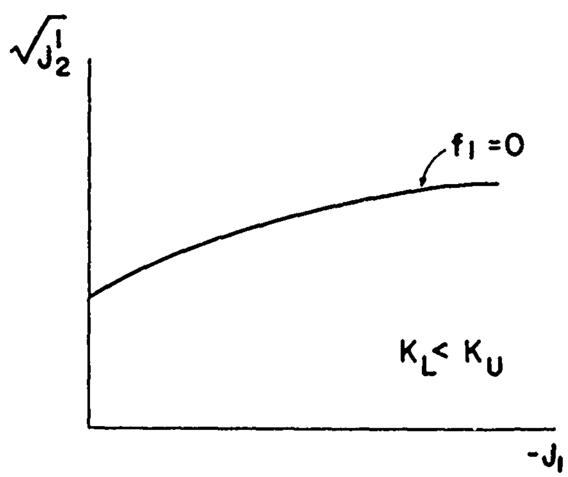

FIG. 6 IDEALLY PLASTIC MODEL (WITHOUT CAP) WITH DIFFERENT BULK MODULI ON LOADING AND UNLOADING

APPENDIX B

CAP MODEL BEHAVIOR IN UNIAXIAL STRAIN

The general material model considered here can be described in three dimensional Cartesian coordinates by the incremental relations

$$d\varepsilon_{ij} = d\varepsilon^E_{ij} + d\varepsilon^P_{ij} \tag{B-1}$$

$$d\varepsilon^E_{ij} = \frac{1}{9K}\,\delta_{ij}\;d\sigma_{kk} + \frac{1}{2G}\,[d\sigma_{ij} - \frac{1}{3}\,\delta_{ij}d\sigma_{kk}] \tag{B-2}$$

$$F(\sigma_{ij}\;,\;\varepsilon^P_{ij}) \leq 0 \tag{B-3}$$

$$d\varepsilon^P_{ij} = \begin{cases} d\lambda\,\dfrac{\partial F}{\partial \sigma_{ij}} & \text{if } F = 0 \\ 0 & \text{if } F < 0 \end{cases} \tag{B-4}$$

in which the summation convention has been adopted, δ_{ij} is the Kronecker delta, and the $d\varepsilon_{ij}$, $d\varepsilon^E_{ij}$ and $d\varepsilon^P_{ij}$ denote the increments of total strain, elastic strain, and plastic strain, respectively. The $d\sigma_{ij}$ denote the stress increments, and $d\lambda$ is a coefficient which is non-zero only when plastic deformations occur. During plastic deformation Eqs. (B-3, 4) become

$$F(\sigma_{ij}\;,\;\varepsilon^r_{ij}) = 0 \tag{B-5}$$

$$d\varepsilon^P_{ij} = d\lambda\,\frac{\partial F}{\partial \sigma_{ij}} \tag{B-6}$$

The elastic and plastic strains can be eliminated from Eqs. (B-1, 2, 3, 4) by differentiating Eq. (B-5)

$$\frac{\partial F}{\partial \sigma_{ij}}\;d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon^P_{ij}}\;d\varepsilon^P_{ij} = 0 \tag{B-7}$$

and substituting Eq. (B-2, 6) into Eqs. (B-1, 7) to obtain

$$d\varepsilon_{ij} = (\frac{1}{9K} - \frac{1}{6G})\delta_{ij}\, d\sigma_{kk} + \frac{1}{2G}\, d\sigma_{ij} + d\lambda\frac{\partial F}{\partial\sigma_{ij}} \tag{B-8}$$

$$\frac{\partial F}{\partial\sigma_{ij}}\, d\sigma_{ij} + d\lambda\frac{\partial F}{\partial\sigma_{ij}}\,\frac{\partial F}{\partial\varepsilon^p_{ij}} = 0 \tag{B-9}$$

Multiplication of Eq. (B-8) by δ_{ij} gives

$$d\sigma_{kk} = 3K(d\varepsilon_{kk} - d\lambda\,\frac{\partial F}{\partial\sigma_{rs}}\,\delta_{rs}) \tag{B-10}$$

and substitution of Eq. (B-10) into Eq. (B-8) leads to

$$\begin{aligned} d\sigma_{ij} = {} & 2Gd\varepsilon_{ij} + (K - \frac{2G}{3})\delta_{ij}(d\varepsilon_{kk} - d\lambda\frac{\partial F}{\partial\sigma_{rs}}\,\delta_{rs}) \\ & - 2G\, d\lambda\frac{\partial F}{\partial\sigma_{ij}} \end{aligned} \tag{B-11}$$

Multiplication of Eq. (B-11) by $\frac{\partial F}{\partial\sigma_{ij}}$ and subtraction of the resulting relation from Eq. (B-9) gives an equation which may be solved for $d\lambda$

$$d\lambda = \frac{(\frac{\partial F}{\partial\sigma_{rs}})[(K - \frac{2G}{3})\delta_{rs}\, d\varepsilon_{kk} + 2G\, d\varepsilon_{rs}]}{(K - \frac{2G}{3})(\frac{\partial F}{\partial\sigma_{rs}}\,\delta_{rs})^2 + 2G(\frac{\partial F}{\partial\sigma_{rs}})^2 - \frac{\partial F}{\partial\sigma_{rs}}\,\frac{\partial F}{\partial\varepsilon^p_{rs}}} \tag{B-12}$$

This may be rewritten as

$$d\lambda = \frac{(\frac{\partial F}{\partial\sigma_{rs}})[Kd\varepsilon_{KK}\delta_{rs} + 2G\, de_{rs}]}{K(\frac{\partial F}{\partial\sigma_{rs}}\,\delta_{rs})^2 + 2G(\frac{\partial F}{\partial\sigma_{rs}} - \frac{1}{3}\,\frac{\partial F}{\partial\sigma_{pq}}\,\delta_{pq}\delta_{rs})^2 - \frac{\partial F}{\partial\sigma_{rs}}\,\frac{\partial F}{\partial\varepsilon^p_{rs}}} \quad \text{(B-1}$$

in which $de_{rs} = d\varepsilon_{rs} - \frac{1}{3}\, d\varepsilon_{kk}\delta_{rs}$ are the deviatoric strains. The incremental stress strain relations during plastic deformation may be obtained by substituting Eq. (B-13) into Eq. (B-11). The yield surface $F(\sigma_{ij}\ ,\ \varepsilon^p_{ij}) = 0$ in the models

considered here are assumed to involve only the invariants

$$J_1 = \sigma_{ij}\, \delta_{ij} = \sigma_{kk} \tag{B-14}$$

$$J_2' = \frac{1}{2} S_{ij}S_{ij} = \frac{1}{2}(\sigma_{ij} - \frac{1}{3} J_1\delta_{ij})(\sigma_{ij} - \frac{1}{3} J_1\delta_{ij}) \tag{B-15}$$

$$\varepsilon_v^p = \varepsilon_{rs}^p\, \delta_{rs} = \varepsilon_{kk}^p \tag{B-16}$$

in which the S_{ij} are the deviatoric stresses. Therefore

$$F(J_1, \sqrt{J_2'}, \varepsilon_v^p) = 0 \tag{B-17}$$

and

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial J_1} \frac{\partial J_1}{\partial \sigma_{ij}} + \frac{1}{2\sqrt{J_2'}} \frac{\partial F}{\partial \sqrt{J_2'}} \frac{\partial J_2'}{\partial \sigma_{ij}} = \frac{\partial F}{\partial J_1} \delta_{ij} + \frac{\partial F}{\partial \sqrt{J_2'}} \frac{s_{ij}}{2\sqrt{J_2'}} \tag{B-18}$$

$$\frac{\partial F}{\partial \varepsilon_{rs}^p} = \frac{\partial F}{\partial \varepsilon_v^p} \frac{\partial \varepsilon_v^p}{\partial \varepsilon_{rs}^p} = \frac{\partial F}{\partial \varepsilon_v^p} \delta_{rs} \tag{B-19}$$

Introducing Eqs. (B-18, 19) into Eqs. (B-11, 13) gives

$$d\sigma_{ij} = K\delta_{ij}(d\varepsilon_{kk} - 3d\lambda \frac{\partial F}{\partial J_1}) + 2G(de_{ij} - d\lambda\frac{\partial F}{\partial \sqrt{J_2'}} \frac{s_{ij}}{2\sqrt{J_2'}}) \tag{B-20}$$

$$d\lambda = \frac{3 \frac{\partial F}{\partial J_1} Kd\varepsilon_{kk} + G \frac{\partial F}{\partial \sqrt{J_2'}} \frac{s_{rs}}{\sqrt{J_2'}} de_{rs}}{9K(\frac{\partial F}{\partial J_1})^2 + G(\frac{\partial F}{\partial \sqrt{J_2'}})^2 - 3\frac{\partial F}{\partial J_1} \frac{\partial F}{\partial \varepsilon_v^p}} \tag{B-21}$$

For the case of uniaxial strain in the z-direction, the following relations hold

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \tag{B-22}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_r \tag{B-23}$$

Denoting the axial stress and strain by σ_z and ε_z ,

respectively, one can write

$$d\varepsilon_{kk} = d\varepsilon_z \tag{B-24}$$

$$de_{zz} = \frac{2}{3}\, d\varepsilon_z \tag{B-25}$$

$$de_{xx} = de_{yy} = -\frac{1}{3}\, d\varepsilon_z \tag{B-26}$$

$$de_{xy} = de_{yz} = de_{zx} = 0 \tag{B-27}$$

$$J_1 = \sigma_z + 2\sigma_r \tag{B-28}$$

$$s_{zz} = \frac{2}{3}(\sigma_z - \sigma_r) \tag{B-29}$$

$$s_{xx} = S_{yy} = -\frac{1}{3}(\sigma_z - \sigma_r) \tag{B-30}$$

$$s_{xy} = S_{yz} = S_{zx} = 0 \tag{B-31}$$

$$\sqrt{J_2'} = \frac{1}{\sqrt{3}}\,|\sigma_z - \sigma_r| = \frac{\sqrt{3}}{2}\,|s_z| \tag{B-32}$$

Substitution of these equations into Eq. (B-20, 21) gives

$$d\lambda = \frac{\left(3K\frac{\partial F}{\partial J_1} + G\frac{\partial F}{\partial\sqrt{J_2'}}\;\frac{2s_z}{\sqrt{3}\,|s_z|}\right) d\varepsilon_z}{9K\left(\frac{\partial F}{\partial J_1}\right)^2 + G\left(\frac{\partial F}{\partial\sqrt{J_2'}}\right)^2 - 3\frac{\partial F}{\partial J_1}\;\frac{\partial F}{\partial\varepsilon_v^p}} \tag{B-33}$$

and

$$ds_z = 2G\left(\frac{2}{3}\, d\varepsilon_z - d\lambda\,\frac{\partial F}{\partial J_2'}\, s_z\right) \tag{B-34}$$

$$dJ_1 = 3K\left(d\varepsilon_z - 3d\lambda\,\frac{\partial F}{\partial J_1}\right) \tag{B-35}$$

Furthermore,

$$d\varepsilon_v^p = d\lambda\,\frac{\partial F}{\partial\sigma_{ij}}\,\delta_{ij} = 3d\lambda\,\frac{\partial F}{\partial J_1} \tag{B-36}$$

Equations (B-33, 34, 35, 36) may be integrated numerically to obtain the stress-strain curve of the model in uniaxial strain provided that Eq. (B-33) can be evaluated for $d\lambda$. This can always be done except when

$$\frac{\partial F}{\partial J_1} = \frac{\partial F}{\partial \sqrt{J_2'}} = 0 \tag{B-37}$$

which can happen for the model of Section IV only for $F = F_2$ if and only if

$$s_z = J_1 = \varepsilon_v^p = L = X = 0 \tag{B-38}$$

In this case one may employ an asymptotic approach in which all quantities which vanish are expanded about the origin in stress space

$$J_1 = \dot{J}_1 \, t \quad , \quad dJ_1 = \dot{J}_1 \, dt \tag{B-39}$$

$$\varepsilon_z = \dot{\varepsilon}_z \, t \quad , \quad d\varepsilon_z = \dot{\varepsilon}_z \, dt \tag{B-40}$$

$$\lambda = \dot{\lambda} \, t \quad , \quad d\lambda = \dot{\lambda} \, dt \tag{B-41}$$

$$s_z = \dot{s}_z \, t \quad , \quad ds_z = \dot{s}_z \, dt \tag{B-42}$$

$$\varepsilon_v^p = \dot{\varepsilon}_v^p \, t \quad , \quad d\varepsilon_v^p = \dot{\varepsilon}_v^p \, dt \tag{B-43}$$

where t is a parameter which is to approach zero.

Then

$$\frac{\partial F_2}{\partial J_1} = 2(J_1 - L) = 2(\dot{J}_1 - \dot{L})t \tag{B-44}$$

$$\frac{\partial F_2}{\partial \sqrt{J_2'}} = 2R^2 \sqrt{J_2'} = R^2 \sqrt{3} \, |s_z| \tag{B-45}$$

Substituting Eqs. (B-39 to 45) into Eqs. (B-33 to 36) gives

$$\dot{\lambda}t = \frac{[6K_i(\dot{J}_1 - \dot{L}) + 2G_iR_i^2\,\dot{s}_z]\dot{\varepsilon}_z}{36K_i(\dot{J}_1 - \dot{L})^2 + 3G_iR_i^4\,\dot{s}_z^2 - 6(J_1 - L)\,\frac{\partial F}{\partial \varepsilon_v}\,\frac{1}{t}} \tag{B-46}$$

$$\dot{s}_z = 2G_i(\frac{2}{3}\,\dot{\varepsilon}_z - \dot{\lambda}R_i^2\,\dot{s}_z t) \tag{B-47}$$

$$\dot{J}_1 = 3K_i[\dot{\varepsilon}_z - 6\dot{\lambda}\,(\dot{J}_1 - \dot{L})t] \tag{B-48}$$

$$\dot{\varepsilon}_v^p = 6\dot{\lambda}\,t\,(\dot{J}_1 - \dot{L}) \tag{B-49}$$

in which K_i , G_i and R_i are the bulk modulus, shear modulus and cap shape factor under conditions (B-38).

Consider Eq. (16) of Section IV,

$$\ell - X = R[A - Ce^{B\ell}] \tag{B-50}$$

By means of Eqs. (22, 24) it is clear that $\ell > X$, which implies that $\ell > 0$. Therefore, by Eq. (16), $\dot{L}$ is instantaneously equal to zero, even though $\dot{\ell}$ is not zero. For $\dot{L} = 0$

$$\begin{aligned}\frac{\partial F_2}{\partial \varepsilon_v^p} &= \frac{\partial F_2}{\partial x}\,\frac{dX}{d\varepsilon_v^p} = -2(X - L)\frac{dX}{d\varepsilon_v^p} = -2\dot{X}t\,\frac{dX}{d\varepsilon_v^p} \\ &= -2t\left(\frac{dX}{d\varepsilon_v^p}\right)^2 \dot{\varepsilon}_v^p = -12\dot{\lambda}t^2\,\dot{J}_1\left(\frac{dX}{d\varepsilon_v^p}\right)^2\end{aligned} \tag{B-51}$$

Then Eq. (B-46) becomes

$$\dot{\lambda}t = \frac{(6K_i\dot{J}_i + 2G_iR_i^2\,\dot{s}_z)\,\dot{\varepsilon}_z}{36K_i\dot{J}_i^{\,2} + 3G_iR_i^4\,\dot{s}_z^2 + 72\,\dot{J}_1^{\,2}\left(\frac{dX}{d\varepsilon_v^p}\right)^2\dot{\lambda}t} \tag{B-52}$$

Equation (B-61) is useful for fitting the model in the low stress region where the seismic behavior of the material may be important.

Let

$$s = \frac{\dot{s}_z}{\dot{\varepsilon}_z} , \quad J = \frac{\dot{J}_1}{\dot{\varepsilon}_z} , \quad \Lambda = \dot{\lambda} t , \quad Q = \frac{dX}{d\varepsilon_v^p} \tag{B-53}$$

Then Eqs. (B-47, 48, 52) become

$$s = \frac{4G_i}{3} - 2G_i \, \Lambda \, R_i^2 \, s \tag{B-54}$$

$$J = 3K_i - 18 \, K_i \, \Lambda \, J \tag{B-55}$$

$$\Lambda = \frac{6K_i J + 2G_i R_i^2 s}{36K_i J^2 + 3G_i R_i^4 \, s^2 + 72 \, J^2 \, \Lambda \, Q^2} \tag{B-56}$$

Equations (B-54, 55, 56) may be solved for s, J and Λ .

After some algebra one finds

$$\Lambda = \frac{1}{6} \; \frac{\left(\frac{3\xi}{G_i} - \frac{4}{3K_i}\right)}{(4 - R_i^2 \xi)} \tag{B-57}$$

$$J = \frac{3K_i}{1 + 18 \, K_i \Lambda} \tag{B-58}$$

$$s = \xi J \tag{B-59}$$

in which $\xi = s/J$ is the slope of the stress path and satisfies the equation

$$\sqrt{3} \left(\frac{3\xi}{G_i} - \frac{4}{3K_i}\right) \frac{dx}{d\varepsilon_v^p} = (4 - \xi R_i^2) \sqrt{1 + \frac{3}{4} R_i^2 \, \xi^2} \tag{B-60}$$

The initial modulus M_i for uniaxial strain is given by

$$M_i = \frac{\dot{\sigma}_z}{\dot{\varepsilon}_z} = \frac{\dot{s}_z + \frac{1}{3} \dot{J}_1}{\dot{\varepsilon}_z} = s + \frac{1}{3} J = (\xi + \frac{1}{3}) J$$

$$= \frac{3(\xi + \frac{1}{3}) \; K_i (4 - R_i^2 \, \xi)}{4 - R_i^2 \xi + 3K_i \left(\frac{3\xi}{G_i} - \frac{4}{3K_i}\right)} = \frac{(\xi + \frac{1}{3})(4 - R_i^2 \xi)}{\left(\frac{3}{G_i} - \frac{R_i^2}{3K_i}\right)\xi} \tag{B-61}$$

Equation (B-61) is useful for fitting the model in the low stress region where the seismic behavior of the material may be important.